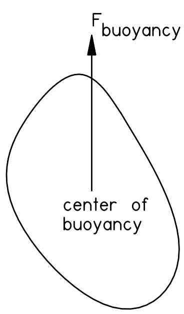
Exploring Buoyancy

Robert L. Belleville

Abstract

Air's density changes pendulum period because buoyancy can slightly reduce gravity's restoring force. Daily changes in temperature, barometric pressure, and relative humidity all change air's density and therefore clock rate. Methods are given to compute air density from environmental measurements and to compute the magnitude of the effect in most pendulums. A design for a pendulum free of buoyancy error is given. Real environmental data is applied to a hypothetical clock to show rate variation.



1 Buoyancy

Buoyancy is an upward force on a body equal to the weight of fluid displaced by that body. This is *Archimedes' principle*. This force acts through a *center of buoyancy* which is the center of mass of the *displaced fluid*.

$$F_{buoyancy} = \rho_{air} V_{pendulum} g \tag{1}$$

where: ρ_{air} is air's density in $kilograms/meter^3$, $V_{pendulum}$ is the pendulum's volume in m^3, g is gravitational acceleration in m/sec^2.

2 Density of Air

Temperature, barometric pressure and the amount of water vapor in air all effect air's density. The relationship is complex.

$$\rho_{air} = \frac{P_{dry\ air}}{R_{dry\ air} T_{abs}} + \frac{P_{water\ vapor}}{R_{water\ vapor} T_{abs}} \tag{2}$$

where: $P_{dry\ air}$ is the pressure of dry air in $Pascals$, $P_{water\ vapor}$ is the pressure of water vapor in $Pascals$, $R_{dry\ air}$ is the gas constant of dry air in $Joules/kg\,^{\circ}Kelvin$ and has the value 287.05, $R_{water\ vapor}$ is the gas constant of water vapor in $Joules/kg\,^{\circ}Kelvin$ and has the value 461.495, T_{abs} is absolute temperature in $^{\circ}Kelvin$—add 273.15 to $^{\circ}C$ to get $^{\circ}Kelvin$.

My laboratory equipment measures relative humidity which is then used to determine $P_{water\ vapor}$. This is accomplished by using the saturation vapor pressure of water which is a function of temperature. P_{vsat} will denote this value. Commonly, relative humidity (RH) is expressed in % and is given by the formula:

$$RH = 100\% \frac{P_{water\ vapor}}{P_{vsat}} \tag{3}$$

Next the saturation vapor pressure of water is given by the program:

```
def saturationVaporPressure( T ):
    """
    from http://wahiduddin.net/calc/density_altitude.htm
    A very accurate, albeit quite odd looking, formula for
    saturation vapor pressure is a polynomial developed
    by Herman Wobus (ref 2 doc at web site also file:
        schlatter_baker_algorithms.txt
    12/11/2006 checked with calculator at web site 0..30 deg C

    where:  Es = saturation pressure of water vapor, mbar
            T  = temperature, deg C
    """

    eso= 6.1078
    c0 = 0.99999683
    c1 = -0.90826951e-2
    c2 = 0.78736169e-4
    c3 = -0.61117958e-6
    c4 = 0.43884187e-8
    c5 = -0.29883885e-10
    c6 = 0.21874425e-12
    c7 = -0.17892321e-14
    c8 = 0.11112018e-16
    c9 = -0.30994571e-19
    p  = (c0+T*(c1+T*(c2+T*(c3+T*(c4+T*
            (c5+T*(c6+T*(c7+T*(c8+T*(c9))))))))))
    Es = eso /  p**8 # this is p to the 8th power
    return Es
```

Barometric pressure in this article will ordinarily be expressed in *millibars* or *mbar*. Multiply by 100 to get *Pascals*. Finally, using the law of partial pressure:

$$P_{dry\ air} = P_{baro} - P_{water\ vapor} \tag{4}$$

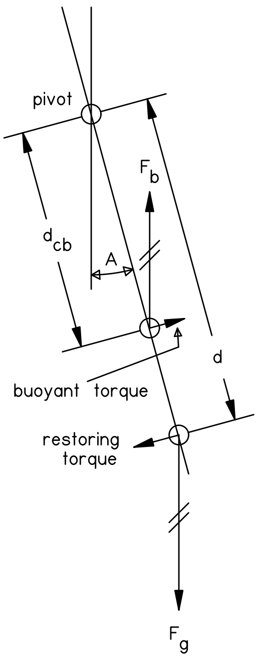
where: P_{baro} is the absolute atmospheric pressure, and all pressures use identical units.

Air has a density in the order of $1.2 kg/m^3$ but it varies quite a bit with temperature, barometric pressure and relative humidity. As this is a four dimensional space, no single plot is of much use. Table 1 shows how each parameter effects density. Also be aware that the density of air *decreases* as the relative humidity *increases*. This is because water vapor molecules H_2O weigh less than the O_2 and N_2 molecules displaced by an equivalent volume of water vapor. Temperature has the largest effect. Barometric pressure's effect, although smaller than temperature's, is much larger than the effect of relative humidity. The range of values for air density is 1.2532 at low temperature and high barometer and 1.1381 at high temperature, low barometer and high RH. The range is $0.1151 kg/m^3$.

Table 1: Density of Air kg/m^3

temp	baro	ρ_{air} for four values of RH			
$°C$	$mbar$	30%	40%	50%	60%
10	1000	1.2286	1.2281	1.2275	1.2269
10	1010	1.2409	1.2404	1.2398	1.2392
10	1020	1.2532	1.2527	1.2521	1.2515
15	1000	1.2067	1.2059	1.2051	1.2043
15	1010	1.2187	1.2180	1.2172	1.2164
15	1020	1.2308	1.2301	1.2293	1.2285
20	1000	1.1852	1.1842	1.1831	1.1821
20	1010	1.1971	1.1961	1.1950	1.1940
20	1020	1.2090	1.2079	1.2069	1.2058
25	1000	1.1642	1.1628	1.1614	1.1601
25	1010	1.1759	1.1745	1.1731	1.1717
25	1020	1.1876	1.1862	1.1848	1.1834
30	1000	1.1436	1.1418	1.1400	1.1381
30	1010	1.1551	1.1533	1.1514	1.1496
30	1020	1.1666	1.1648	1.1629	1.1611

3 Effect on Period



The period of a physical pendulum in a vacuum is given by:

$$t = 2\pi\sqrt{\frac{I}{mgd}} \tag{5}$$

where: t is the period in *seconds*, I is the moment of inertia of the pendulum in $kg\ m^2$, m is the mass of the pendulum in kg, g as before, and d is the distance from center of rotation to the center of mass in m.

The term mgd in equation 5 is the gravitational restoring torque for a pendulum in a vacuum. For most pendulum configurations the center of buoyancy will lie on the line from the center of rotation to the center of mass. If d_{cb} is this distance then:

$$mgd \rightarrow (md - \rho_{air} V_{pendulum} d_{cb})g \tag{6}$$

Given pendulum density ρ_{pend} the volume of any pendulum is:

$$V_{pendulum} = m/\rho_{pend} \tag{7}$$

Combining equations (5), (6) and (7) gives:

$$t = 2\pi\sqrt{\frac{I}{mgd(1 - \frac{\rho_{air}}{\rho_{pend}}\frac{d_{cb}}{d})}} \tag{8}$$

Three interesting cases are seen in equation (8): $d_{cb} = d$, $0 < d_{cb} < d$, and $d_{cb} = 0$.

3.1 $d_{cb} = d$

Buoyancy's effect is reduced in the ratio d_{cb}/d. This ratio is 1 for pendulums of uniform density and will be close to 1 for pendulum configurations with a thin rod and heavy bob as we shall see in the next section.

Setting $d_{cb} = d$ equation (8) becomes:

$$t = 2\pi\sqrt{\frac{I}{mdg(1 - \frac{\rho_{air}}{\rho_{pend}})}} \tag{9}$$

Any effect of ρ_{air} on period will be found from the derivative $dt/d\rho_{air}$:

$$\frac{dt}{d\rho_{air}} = \frac{\pi I}{mgd\left(1 - \frac{\rho_{air}}{\rho_{pend}}\right)^2 \rho_{pend}\sqrt{\frac{I}{mgd\left(1-\frac{\rho_{air}}{\rho_{pend}}\right)}}} \tag{10}$$

Dividing by period creates a normalized equation, $\mathcal{B}$ a 'buoyancy error factor', which effectively removes all parameters except densities and period.

$$\mathcal{B} = \frac{\frac{dt}{d\rho_{air}}}{t} = \frac{1}{2\left(1 - \frac{\rho_{air}}{\rho_{pend}}\right)\rho_{pend}} \tag{11}$$

Since the ratio ρ_{air}/ρ_{pend} is nearly zero equation 11 reduces to:

$$\mathcal{B} = \frac{\frac{dt}{d\rho_{air}}}{t} \approx \frac{1}{2\rho_{pend}} \tag{12}$$

This remarkable result means that $\mathcal{B}$ is solely dependent upon pendulum density. For example, suppose an all-tungsten pendulum has a period of 1 second in vacuum. Tungsten has a density of $19,300\,kg/m^3$. What is the change in period from vacuum at $\rho_{air} = 1.2\,kg/m^3$?

$$\mathcal{B} \approx \frac{1}{38600} = 25.91 \times\ 10^{-6} \tag{13}$$

Multiply (13) by $t = 1sec$ to get $dt/d\rho_{air}$:

$$\frac{dt}{d\rho_{air}} \approx \frac{\triangle t}{\triangle\rho_{air}} = \frac{\triangle t}{1.2} = 25.91 \times\ 10^{-6} \rightarrow\ \triangle t = 31.09 \times\ 10^{-6} seconds \tag{14}$$

Let's check this result with my #8 pendulum which is a simple $7/8\,inch$ rod of brass (C360) about $0.844\,meters$ long. In this case:

$$I \approx \frac{4md^2}{3}\ , l = 2d \tag{15}$$

$$t = 2\pi\sqrt{\frac{2ml}{3(m - \rho_{air}V_{pendulum})g}} \tag{16}$$

$$t = 2\pi\sqrt{\frac{2l}{3(1 - \frac{\rho_{air}}{\rho_{pend}})g}} \tag{17}$$

$$l = \frac{3g(\rho_{pend} - \rho_{air})t^2}{8\pi^2\rho_{pend}} \tag{18}$$

$$V_{pendulum} = \pi r_{cyl}^2 l \tag{19}$$

where: r_{cyl} is the radius of the rod in m and is equal to $0.0111125\,m$. C360 brass has density $8490\,kg/m^3$. Gravitational acceleration at my lab is

$9.799147\,m/s^2$. To get a $1.5\,sec$ period in vacuum, pendulum length would have to be $0.8531804\,m$.

$$\mathcal{B} \approx \frac{1}{16980} = 58.89 \times\, 10^{-6} \tag{20}$$

Multiply (20) by $1.5sec$ to get $dt/d\rho_{air}$. Then use the lower ρ_{air}:

$$\frac{dt}{d\rho_{air}} \approx \frac{\triangle t}{\triangle \rho_{air}} = \frac{\triangle t}{1.1381} = 88.34 \times\, 10^{-6} \rightarrow\ \triangle t = 100.55 \times\, 10^{-6} sec \tag{21}$$

At $\rho_{air} = 1.2532\,kg/m^3$ $t = 110.71 \times\, 10^{-6} sec$. So over the range of air densities in Table 1 there will be a 10.16 microseconds variation in period. This can be confirmed by using equation (17). The results are 100.55 and 110.72 with a range of 10.17—essentially the same.

If we look at variation per day then period also cancels out.

$$\frac{cycles}{day} = \frac{86400}{t};\ \triangle t = t\,\mathcal{B}\,\triangle \rho_{air};\ \triangle t_{day} = 86400\,\mathcal{B}\,\triangle \rho_{air} \tag{22}$$

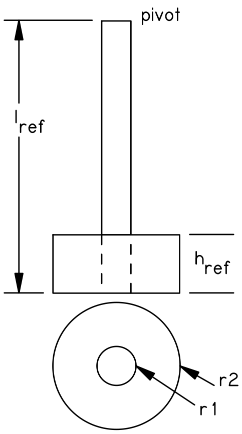
Table 2 shows an uncertainty in daily rate for pendulums built of various materials. Uncertainty here means that if ρ_{air} is unknown then the daily rate will be uncertain over say Table 1's range of densities. This gives a feel for the magnitude of buoyancy error. α, used below, is the coefficient of linear expansion. This is used in the next section.

Table 2: Uncertainty of daily rate due to buoyancy effect

material	density kg/m^3	α per$^\circ C$	$\mathcal{B}$ (buoyancy error factor)	uncertainty sec/day
fused quartz	2,200	0.40×10^{-6}	227.27×10^{-6}	$\pm 1.13 sec$
invar 36	8,050	1.30×10^{-6}	62.11×10^{-6}	$\pm 0.31 sec$
brass c360	8,490	20.5×10^{-6}	58.89×10^{-6}	$\pm 0.29 sec$
tungsten	19,300	4.40×10^{-6}	25.91×10^{-6}	$\pm 0.13 sec$

3.2 $0 < d_{cb} < d$

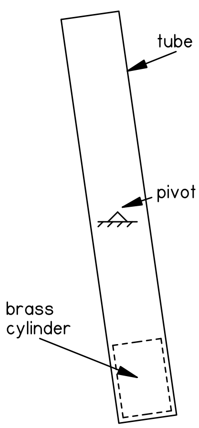
For rod and bob pendulums the ratio d_{cb}/d will vary depending upon the relative mass and density of each part. Table 3 shows six examples of various



pairs of materials. Each pendulum is designed so that the bob is free to move on the rod and is referenced at the bottom of the rod so that it expands upward with temperature just enough to compensate for downward rod expansion. All rods in this table are $1m$ long and pivot from the top. Notice that the heavy tungsten bob on the fourth line has 96% buoyancy error.

Table 3: Ratio d_{cb}/d for temperature compensated pendulums.

rod($r1$) m	bob/($r2$) m	mass bob kg	bob h_{ref} m	d_{cb}/d
invar(0.0015875)	brass(0.030)	3.276	0.136859	0.999
invar(0.0015875)	brass(0.005)	0.172	0.287125	0.995
quartz(0.003)	brass(0.050)	2.704	0.040699	0.97
quartz(0.003)	tungsten(0.020)	4.744	0.020009	0.96
quartz(0.003)	brass(0.020)	0.487	0.046726	0.89
quartz(0.003)	tungsten(0.005)	0.290	0.299792	0.79



3.3 $d_{cb} = 0$

This condition will only be met in compound pendulums where the volume of the upper half is exactly the same as the volume of the lower half. In addition this pendulum must be axially symmetric about a line from the center of rotation to the center of mass. This will place the center of buoyancy exactly coincident with the pivot point. Naturally, the lower half will have to have at least a slightly higher density for the system to oscillate.

One configuration of interest not only eliminates buoyancy error but also can be compensated for error due to thermal expansion. A sealed tube of any low α material which pivots at the exact center has no buoyancy error because the upper and lower halves have the same volume. If a cylinder of higher α material is selected with the correct length and placed at the bottom of this tube then the bottom half has higher density and thermal expansion can also be compensated. A $2m$ tube of fused quartz with an inner radius of $20mm$ and a $2mm$ wall thickness fitted with a $97.021mm$ cylinder of brass with radius $20mm$ (but free to expand upward from the quartz tube's bottom) meets this requirement. It would have a period of 2.33 seconds.

Computations to find dimensions for this type of pendulum are complex. A computer program is available from the author.

Thermally compensated pendulums are subject to transient effects as temperature changes. It is unlikely that rod or tube length changes will exactly track bob changes. Period will vary until temperature stabilizes.

4 Pendulum Volume

Temperature changes the volume of a pendulum by a small amount. How big is the effect on period?

$$\rho_{pend} = \frac{m}{V_{ref}(1 + 3 \bigtriangleup T\alpha)} \tag{23}$$

where: V_{ref} is the pendulum volume at some reference temperature; m is mass as before; $\bigtriangleup T$ is the change in temperature from the reference point; and α is the linear thermal expansion coefficient also called cle. For isotropic materials 3α is a good approximation to the volumetric thermal expansion coefficient.

$$V_{ref} = \frac{m}{\rho_{ref}} \rightarrow \rho_{pend} = \frac{\rho_{ref}}{1 + 3 \bigtriangleup T\alpha} \tag{24}$$

Brass has a large α and over a $10°C$ temperature range, density change is only 0.06%. Most other materials are have a much smaller α.

5 Practical Results

Environmental data is taken within the clock case by a module of my own design. Absolute temperature accuracy is $\pm 0.5°C$ with a resolution of $0.026°C$. Absolute barometric pressure accuracy is $\pm 1.5mbar$ with a resolution of $0.1mbar$. Absolute relative humidity accuracy is $\pm 2\%$ with a resolution of 0.1%.

During 58 logging days in 2006/7 the average air density was 1.206532 but buoyancy error for my brass rod ranged from -0.74 to 0.61 parts per million of period variation. (One ppm = 32 seconds per year.)

Changes in period are substantial. Figure 1 shows actual air density for a 23 day logging run in Nov/Dec 2006 (top). Average density was 1.205383. The lower plot shows rate error for a hypothetical brass pendulum which only responds to buoyancy error—all other changes are set to zero. During

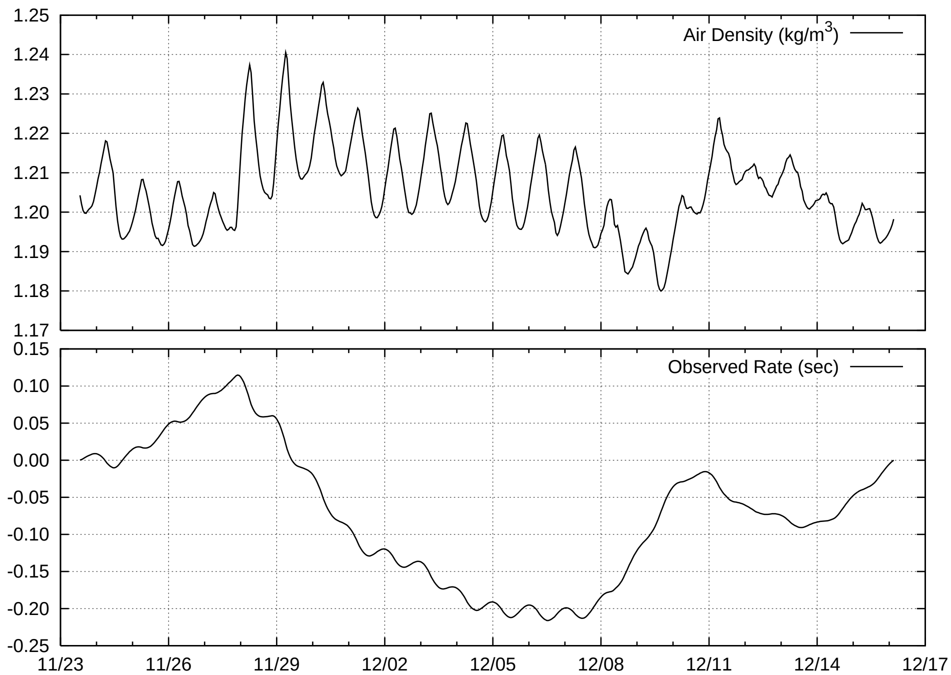


Figure 1: Air Density and Rate Error by Date 2006.

the first five days, lower than average ρ_{air} shortens period and the clock runs faster reaching 0.12 seconds fast. Then for seven days higher than average density increases period and the clock slows to -0.23 seconds slow. For three days density hovers (because of temperature change) about average and a ripple shows in the rate curve. Because this hypothetical clock is normalized for the average density during this run, rate error then returns to zero following variations in density over the remaining time.

6 Conclusion

We have seen that, running in air, most pendulum's period is changed by buoyancy. Buoyancy error is a function of air density and pendulum density. A method of computing air density from temperature, absolute barometric pressure, and relative humidity is given. A table shows air density for a range of environmental conditions.

Air density decreases with increasing temperature and relative humidity and increases with increasing barometric pressure. As air density increases buoyancy increases and period increases. In short, a clock slows as air density increases. ρ_{air} is a complex function of environment, so mechanical compensation would be difficult to achieve.

We have also seen that for a given air density, rate error is purely a function of total pendulum density for most systems.

Buoyancy error can be completely eliminated in compound pendulums where the volume above and below the pivot are identical.

This paper shows that what has historically been called *barometric error* would be more accurately called buoyancy error since the effect is a function not only of barometric pressure but also temperature and relative humidity. Increase in temperature tends to compensate for increase in barometric pressure. The decreased air density from temperature increase tends to make a clock run faster and reduce the need for temperature compensation of pendulum length.

End Note:

To create this document required the use of many programs (most free) including: Python, MS Excel, Maxima, MiKTex, gnuplot, eps2pdf, gsview, ghostscript, TextPad, Adobe Reader, CadStd, and TakeCommand. Many thanks to their authors and maintainers.

Special thanks are due to Martin Haeberli and Ron Crane whose careful reading, verification, and many excellent comments have been essential to me in producing this paper.

Bob Holmström has been especially helpful in the work resulting in this paper. Bob brought Neville Michie's speculation about a buoyancy effect free pendulum to my attention. This resulted in a substantial extension of this paper to clarify how a compound pendulum can eliminate buoyancy error. Many thanks to both Bob and Neville.