Tidal Effects on Pendulums at Various Latitudes

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Abstract

Two independent methods are used to model the change in period of a pendulum located at a given latitude. The average daily effect from the position of the sun and the moon on a pendulum is only zero at latitude 35.2644° (north or south.) Above or below this latitude seasonal variation can cause time variations of many milliseconds. Over the full 19-year cycle of the moon's orbit, time variation could reach hundreds of milliseconds. Both empirical and mathematical methods are used to describe the effect.

1. Introduction

Tom Van Baak's paper *Lunar/Solar Tides and Pendulum Clocks* in HSN 2006-1 introduced the effect of tidal acceleration on pendulum clocks. This paper extends that, and subsequent work, to show the mechanism by which the height (altitude) of both the sun and the moon above the local horizon of a pendulum slightly changes system restoring force. As his paper shows, at any instant the variation in restoring force is exceeding small. It is unlikely that any actual pendulum can be made to show the tidal effect based only on change in pendulum period which is in the order of a few hundred nanoseconds (see below.) However as these pendulum periods are added together to show the time, a ripple in the order of about a millisecond will show up. This might be seen in a super stable pendulum but other system noise is likely to mask this effect too. This paper will show that the average tidal value is only zero near 35.25° latitude (north or south.) So, at higher latitudes, a change of about 10 milliseconds or more could be seen if a pendulum was stable through the seasons as we shall see.

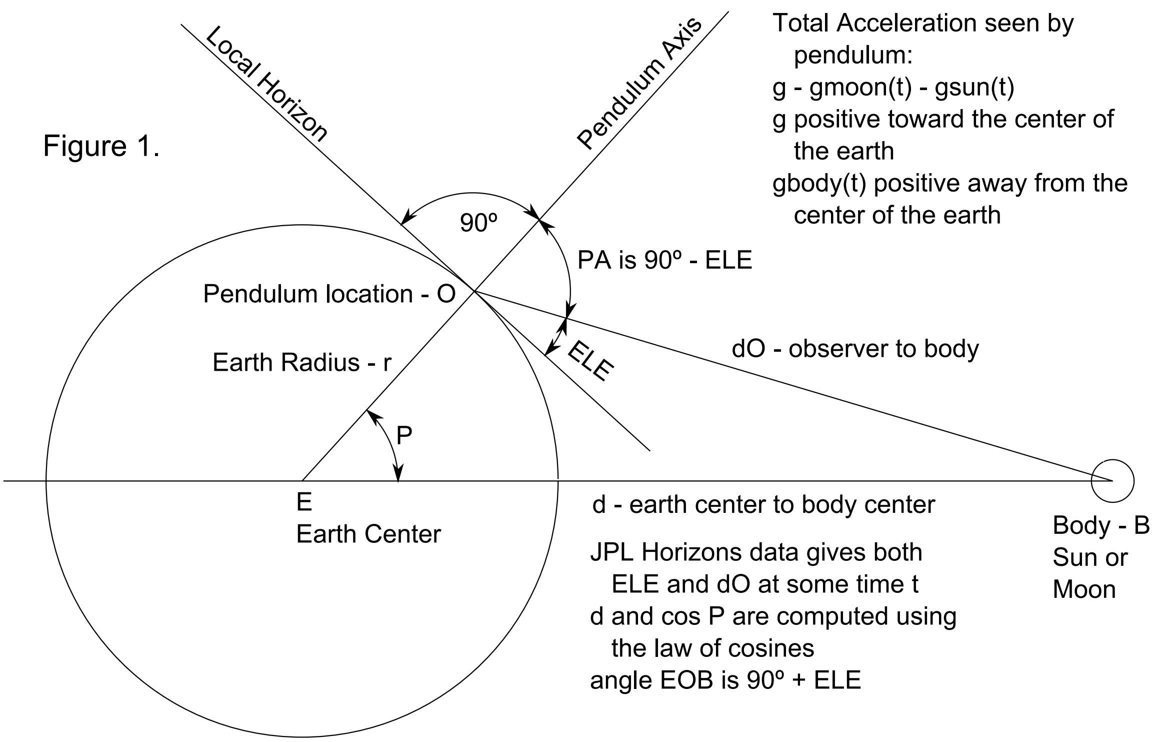
For this paper I will assume a 1 meter simple pendulum which by some magic has constant amplitude and hence constant circular error. The only change in the period will come from change in the gravitational acceleration seen by the pendulum:

$$T = 2\pi\sqrt{\frac{1}{g - g_m - g_s}} \tag{1.1}$$

Here T is the period in seconds which is about 2.007 seconds. Also g, g_m, g_s are the gravitational acceleration of the earth here taken to be 9.8 m/s^2 and the others are for the sun and the moon which are quite small as previously shown. Figure 1 shows the geometry for finding g_m or g_s. The amount of tidal acceleration is the difference between the acceleration of the body on the the pendulum and the acceleration of the body on the center of the earth. Only the vertical component is needed because the horizontal component, while it does pull the pendulum off vertical, is so small that I will ignore it here. Clearly tidal acceleration depends on the distance between the body and the pendulum *dO* and the angle of body above or below the horizon of the pendulum site *ELE*.

All this changes with the rotation of the earth and the orbits of both the sun and the moon. Here *t* represents this time expressed UTC. We will need some ephemeris to give us the distance and elevation. Two different systems will be used here. One, *tides*, includes a built-in ephemeris. The other, *readHorizons*, requires an ephemeris generated by JPL's Horizons online system but which is of the highest accuracy available today.

Figure 1.



Starting from Newton:

$$F = ma, \quad F = \frac{GMm}{d^2} \tag{1.2}$$

where *F* is a force vector, *m* is pendulum mass and *M* the other body, *G* is the gravitational constant, and *d* is the distance between the bodies. We want to get rid of *m* and so we combine the two equations to work in terms of acceleration *a* or in our case *g*.

$$g = \frac{GM}{d^2} \tag{1.3}$$

We are only interested in the vertical acceleration at the pendulum so we resolve the acceleration vector by multipling by *cosPA*. Finally *tidal acceleration* for a rigid earth is given by the acceleration at the pendulum minus the acceleration at the center of the earth.

$$g_{tidal} = \frac{GM}{dO^2}\cos PA - \frac{GM}{d^2}\cos P \tag{1.4}$$

The law of cosines, body *ELE*, and distance *dO* can be used to compute tidal acceleration for the sun or the moon directly but it is hard to visualize the acceleration field from this equation. So we eliminate *PA* and *dO*. Reference 1 shows the relationship of tidal acceleration of the sun or moon for a rigid earth (accuracy of better than 0.05%) to be:

$$d^2 = r^2 + dO^2 - 2r\,dO\cos(90 + ELE) \tag{1.5}$$

$$\cos P = \frac{dO^2 - r^2 - d^2}{-2rd} \tag{1.6}$$

$$g_m = \frac{GM_{moon}r}{d^3}(3\cos^2 P - 1) + \left(\frac{3}{2}\right)\frac{GM_{moon}r^2}{d^4}(5\cos^3 P - 3\cos P) \tag{1.7}$$

$$g_s = \frac{GM_{sun}r}{d^3}(3\cos^2 P - 1) \tag{1.8}$$

References 1, 2, 3 and 4 all show how all these results are derived.

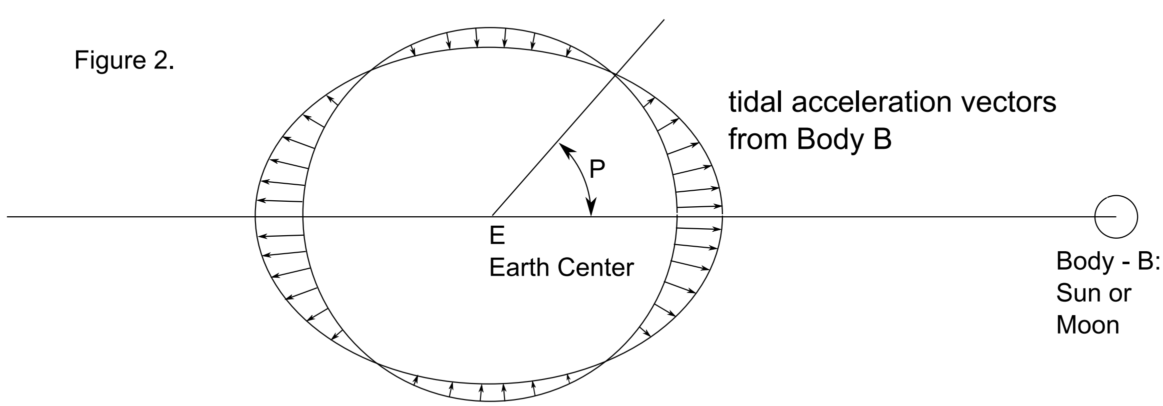
Radius *r* is given by Longman (ref 1) to be:

$$c = \frac{1}{\sqrt{1 + 0.006738\sin Lat}} \tag{1.9}$$

$$r = 6.37827\times 10^8 c + Elevation \tag{1.10}$$

here *r* is in cm. WGS-84 (1984) uses an equitorial radius of 6,378,137 m and 6,356,752.3142 m as the polar radius. If 0.006739497 replaces 0.006738 and WGS-84 equitorial radius is used *r* matches WGS-84's ellipsoid very closely. Reference 7 helps explain that the geocentric radius and the geodetic radius are not the same and shows the size of the error in Lat to be at most about 0.2°. In short at various latitudes a plumb bob doesn't actually point to the center of the earth. I will use Longman and ignore these small errors.

Figure 2.



In this model I use a constant value of g because we are only concerned with variations in g_s and g_m and remove the average value of period in any case. This is also a good place to point out that the average values of g_s and g_m are rarely zero. So in our case the average value of these accelerations are also removed by averaging the period.

The second factor of the first term of g_{moon} is $(3\cos^2 P - 1)$. So at either P=0 or P=180 $\cos^2$ is 1 and the factor is +2. At P=90 cos is 0 and the factor is -1. Only at $cosP = \sqrt{1/3}$ or 54.74° (and the symmetrical points) is the tidal acceleration actually zero. Angles P and PA are always nearly equal because the body is so far away compared to the radius of earth. ELE is 90-PA and approximately 90-P so elevation below angle 90-54.74=35.26° results in a negative acceleration even though the body is still visible and would seem rather high in the sky.

The product GM can be measured astronomically with great accuracy but G or M are known to much lower precision. These *standard gravitational parameters* GM_{sun} and GM_{moon} are given in the header to JPL ephemeris results and are $1.32712440018 \times 10^{11}$ km^3 s^{-2} and 4902.798±0.005 km^3 s^{-2} respectively.

So far we have tidal acceleration for a rigid earth. The earth isn't in fact rigid and there are harmonic vibrations seen at the surface caused by tidal forces on our home sweet home. These tend to increase tidal acceleration actually seen at any point on the globe. References 3 and 4 go into this in some detail. Generally I will use 1.16 from reference 6 as a factor for the observed values of g_m and g_{sun} in effect increasing them by 16%.

Earth varies from around 356,400 km to 406,700 km at the extreme perigees (closest) and apogees (farthest) from the moon. At perihelion, the Earth is about 0.98329 astronomical units (AU) or 147,098,070 km from the sun. The distance of the aphelion is currently about 1.01671 AU or 152,097,700 km. (Reference 9)

So the maximum g_{sun} at the equator and sun directly overhead (cos = 1) when the sun is closest to earth is:

$$g_s = \frac{2GM_{sun}r}{d^3} = 5.3188\times10^{-7}(m/s^2) \tag{1.11}$$

moon:

$$g_m = \frac{2GM_{moon}r}{d^3} + \frac{3GM_{moon}r^2}{d^4} = \frac{2GM_{moon}r}{d^3}\left(1 + \frac{3r}{2d}\right) = 1.4234\times10^{-6}(m/s^2) \tag{1.12}$$

The second order term contributes a bit less that 2.7% of the total and can never be more that this. These maximum values are subtracted from g and make the clock slower. Minimum values are found at cos = 0. The second term for the moon is now zero and -1 replaces 2 in the equations. For the moon the minimum is $-6.9309\times10^{-7}(m/s^2)$ and $-2.6594\times10^{-7}(m/s^2)$ for the sun. These are rigid earth numbers and need to be increased by 16% to give their total effect.

2. Getting Tidal Data

A line from the JPL Horizons ephemeris for the new moon on Oct 26, 2011 at my location Lat 37.365°N Long 122.083°W is:

```
Date        Hour  vv  Azimuth   ELE     dO (in AU)         delta dO
2011-Oct-26 00:00 *m  251.3203   3.5630 0.00238539911047   0.3280682
```

On this day the moon was as high as 36° above my horizon and as low as -66° below tracing out a graceful sine curve. One AU is 149597870.691 km. Just ignore vv and delta dO fields as we don't need them. We don't need Azimuth either as we are ignoring any horizontal component of *g*.

To use JPL Horizons go to http://ssd.jpl.nasa.gov/horizons.cgi#top and use the change buttons to set:

set Ephemeris Type to OBSERVER

set Target Body to Moon [Luna] [301] or Sun [Sol] 10

set Observer Location to pendulum latitude, longitude, and elevation

set Time Span as needed

set Table Settings via check boxes 4 and 20 only

set Display/Output to download/save

Then press the *Generate Ephemeris* button to create a file. A cookie on your computer saves this information for the next request. A file 'horizons_results.txt' will be stored in the download directory used by your web browser. See reference 10 for the Horizons documentation and accuracy.

Downloaded files are all given the same name by Horizons and need to be renamed to be used because at least one ephemeris each for the sun and moon are needed. Program *readHorizons* is available to process the files downloaded by Horizons. For *readHorizons* don't check the CSV format checkbox in 'Table Settings'. My program has several options to produce output for further processing and spreadsheet use. (Contact me for a copy.)

Longman's equations for tidal acceleration and the position of the sun and the moon were captured in a Basic Language program by J. L. Ahern in 1993. (Reference 8) Tom Van Baak translated this program to the C Language. It contains a set of equations needed to compute the terms d and $\cos P$ as well as r for a given latitude. It is fast and easy to use. I modified the program to remove a 12 hour error introduced in the translation from Basic to C and added a number of output formats.

3. Verifing the Software

With JPL Horizons it is safe to assume that the results are as correct as modern technology can provide and that many people are dependent on correct results. But the system is complex and it is a good idea to have a confidence test to be sure we know how to get the actual data we want and to see that it at least matches what we see out the window.

By measuring the angle 22° cast by a pin on a board at 17H UTC on Nov 7, 2011 at (37.365 -122.084) and comparing it to the result given by Horizons of 22.7871° for the sun at that date and hour I concluded that JPL data was as expected.

readHorizon was tested to be sure *ELE* and *dO* were read correctly from the JPL input.

Computing g_{body} is quite straight forward using equations 1.4 to 1.6 above. Equations 1.7 and 1.8 are derived from a series expansion and are quite accurate and used in *tides* but are not used in *readHorizons.* Next *tides* and *readHorizons* output were compared for a long sample of data from Jul 1, 2011 to Dec 30, 2012 some 549 days with hourly samples for a total of 13176 comparisons.

The results are below. Given the difficulty of predicting the moon with accuracy the agreement is remarkable.

	moon μGal	sun μGal
average difference	-0.0082	0.0011
minimum difference	-3.236	-0.104
maximum difference	2.928	0.0646

Finally reference 11 describes an accurate gravitometer built at Stanford University. A nine day run is plotted in the paper and I used readHorizon to show gravity at that laboratory and during that time in early 1997. To a best visual approximation both JPL and *tides* match their result.

So although all times and locations cannot be checked, I am reasonably confident that the software presents correct results.

4. Two days near new moon Sept 2011

Using JPL Horizons and readHorizons an ephemeris for the sun and the moon were created for two days centered on a new moon:

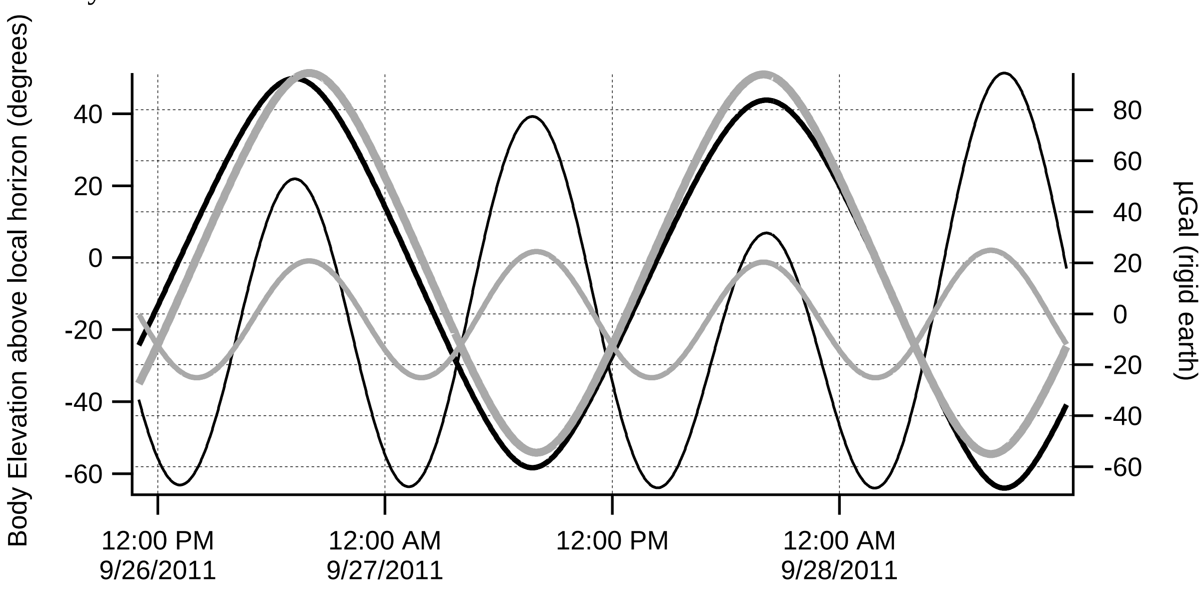


Chart 1a. Two days with New Moon at 9/27/2011 11:09 UTC (near center)

Near new moon both sun and moon appear together in the sky, so Chart1a shows both elevations together (moon heavy black, sun heavy grey.) The range is about +50° to -60°. Note that the tidal acceleration (here in microGals) oscillates twice as fast as elevation. Figure 2 above shows why that is so. The curves are moon in thin black and the sun in thin grey. Also notice how each body has to be well above or below the local horizon before the acceleration is positive (up) thus making the period longer (clock slower.) This is because tidal acceleration (positive up) is subtracted from earth acceleration – less *g* slower, more *g* faster.

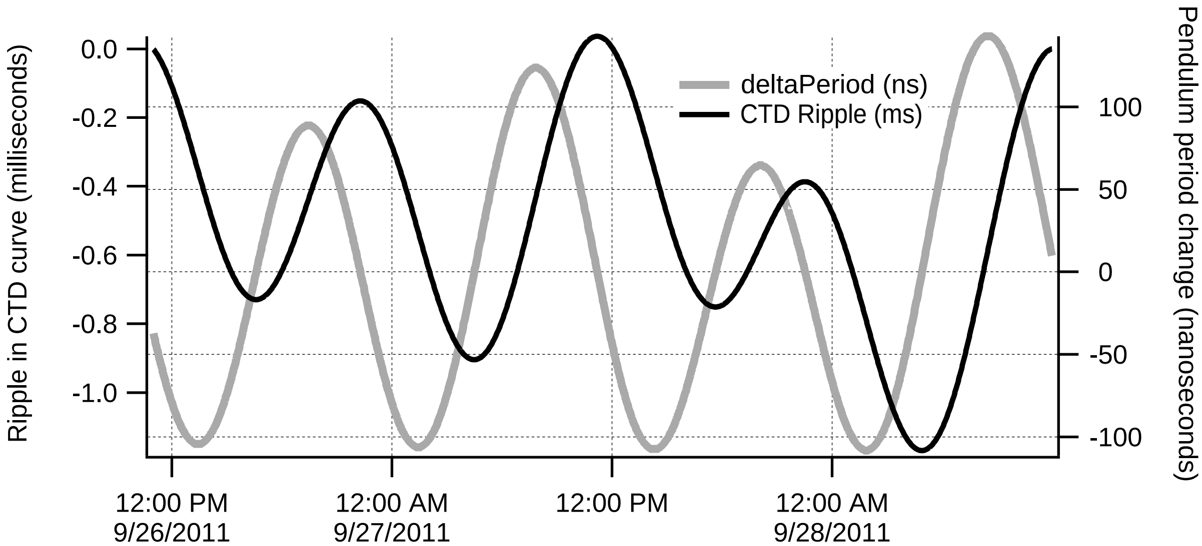


Chart 1b.

Next in Chart1b we have the actual variation in pendulum period here shown in grey in a nanosecond scale. The variation of ±100ns is likely too small to measure in a real clock. But, and this is the main point of this paper, if we add up these small variations from the average period we see a ripple in the cumulative time deviation (CTD) curve (which should be just a flat line) of -1.2ms. This is due to the constantly changing position of the sun and moon over days, seasons, years and latitude of the clock. This CTD adds up the differences between actual period at each instant of time and the mean period for the whole interval. Appendix A details the computation and discusses some of the side effect of this method of analysis.

5. Two Years and Eight Latitudes

JPL ephemeris data for latitudes 0° to 90° in steps of 15° plus latitude 35.25° for both the sun and moon were created. The longitude was -122.084° at sea level. One of the modes of *readHorizons* was used to process all of this data to create a huge table of tidal acceleration and CTD ripple at each position and each hour for two years. Chart 2 shows CTD ripple at three latitudes 30°, 35.25°, and 45° for 2 years on a millisecond scale.

Chart 3 summarizes all this data. On the top trace (diamond markers) notice that average tidal acceleration is positive and highest at the equator and drops off to zero at about 35.25°. Then above that latitude it is negative and reaches its lowest at the pole of less than -60 µGal.

CTD ripple range (circle markers) also change with latitude. At the equator the range is 16 ms and tends negative with the seasons. At 35.25° the range is at a minimum of 2 ms and has no seasonal trend. Above this latitude the range increases again to 10 ms at 45° and has a maximum of 32 ms at the pole.

To emphasize, to get the CTD ripple the period of our hypothetical clock (equation 1.1) is computed using the total gravitational acceleration seen by the pendulum. This is earth's assumed constant pull plus or minus tidal acceleration from the sun and moon. If these bodies are high in the sky, either above the horizon or far below it, the pull is upward and tends to cancel out some of earth's pull – g gets smaller period gets larger and the clock seems to run slow. With the body near the horizon the effect is reversed. Once all these period values are computed for each latitude, the average period is computed. Next 3600 (because the samples are hourly) is divided by the average period to get the number of cycles per hour of the clock. This number, which is just less than 1800, is multiplied by the current clock period minus the average to give an approximation of the period of the clock that hour. Starting at zero, each of these small differences from the mean clock period are added one by one to give Chart 2 shown. If the clock is in a season where tidal acceleration is a bit higher than average, the CTD ripple curve will trend down because the clock will be slow compared to average.

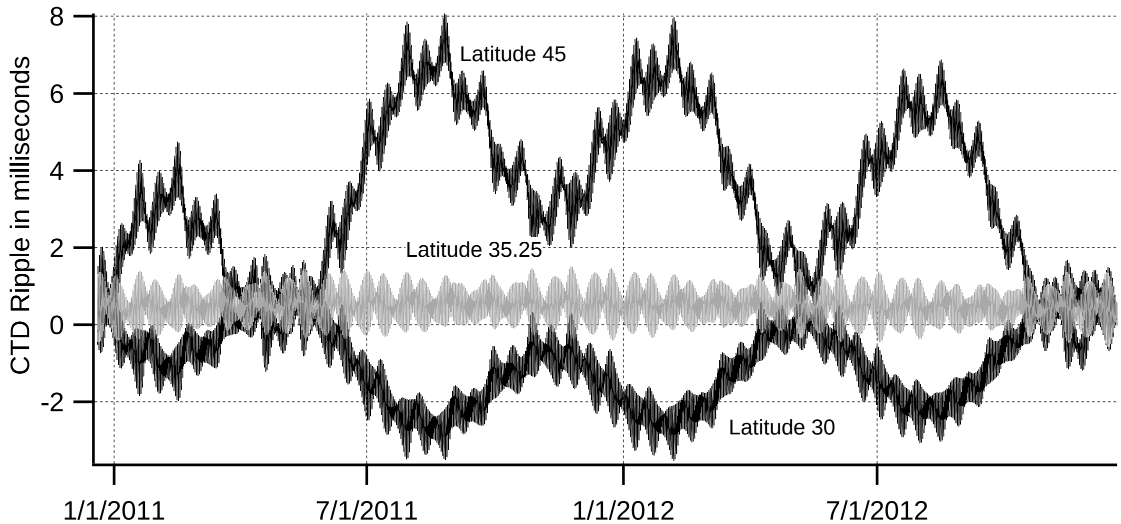


Chart 2. Dec 20, 2010 to Dec 20, 2012

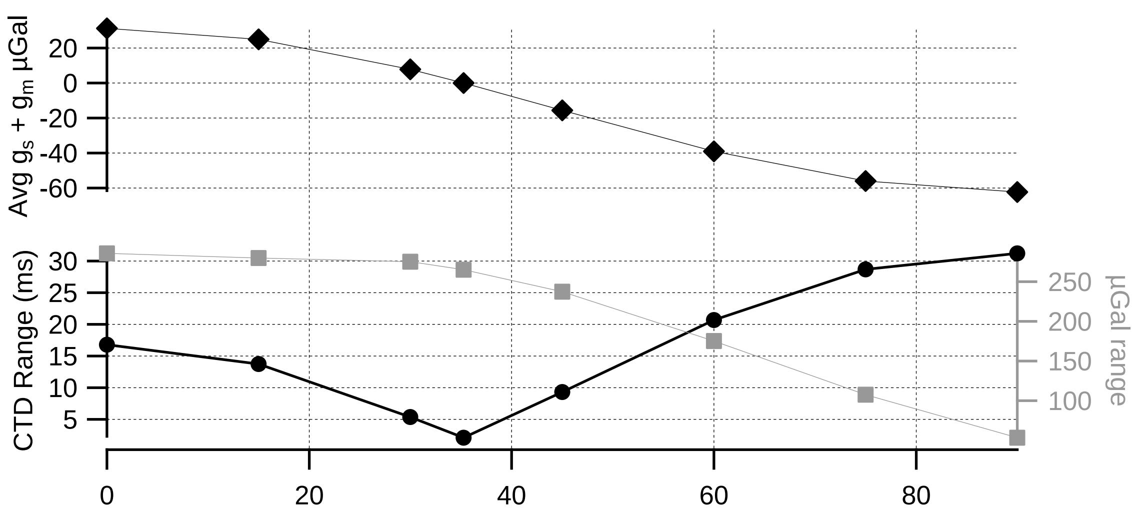


Chart 3. Gravity results by Latitude

In chart 2 the seasonal changes are clearly due to the sun. In chart 3 the trace with diamond markers cleary shows that the average tidal acceleration is greater below 35.25° and less above that latitude. This means that, in addition to the ripple, clocks below 35.25° actually run a little slower while ones above run faster. This is not the same effect as the change in *g* due to latitude because of the ellipsoidal shape of the earth. The elliptical orbit of the earth with the sun closer to the earth in the northern hemisphere winter explains part of this variation.

In chart 2 for the ripple at 35.25°, the effect of the sun is quite small and little yearly trend can be seen. The moon's distance from the earth varies a good deal but the average effect is small over years.

6. Analytic Exploration of Longer Periods

To extend our understanding of these effects we need to move from empirical work to a more analytical approach. A widely-used method of mapping celestial objects is the *equatorial coordinate system.* Here the sky is mapped much like the earth using the earth's axis and equator as reference. In this system *declination* is used like latitude to give a body's angle above or below the equator. Longitude is given by *right ascension* referenced not to Greenwich but to the vernal equinox point. To relate right ascension to a given longitude one needs the time and date to find the *hour angle.* Any number of references explain this is a good deal more detail than is warrented here (reference 12 for a start). Spherical trig gives us a way to convert equatorial coordinates to angle *P. (Reference* 13, ref 1 eqn 15, and ref 2 eqn 31.)

$$\cos P = \sin Lat \sin Dec + \cos Lat \cos Dec \cos(HA - 180°) \tag{6.1}$$

In a year, the declination of the sun moves slowly from -23.44° (about) at the winter solstice to 0 at vernal equinox to 23.44° at summer solstice and then to 0 at autuminal equinox before repeating the cycle each year. So on any given day the value of the sun's declination doesn't actually change much.

The moon's orbit is inclined to the plane of the earth's orbit about the sun by 5.145°. Each month the moon declination will be ±5.145° from the declination of the sun. The moon's maximum declination of 28.585° can only happen when both the sun's maximum declination (northern hemisphere summer soltice) and the moon reaches its maximum angle of 5.145° from the ecliptic plane at the same time. This is rare. (Reference 14.) There are days when the declination of the moon changes as much as 4° but ordinarily the number is much smaller. It is easy to get declination and right ascension from JPL by switching item 4 in table settings to item 1.

Note that the average declination of the sun over a year is zero. Over a single orbit the average declination of the moon will be about the average declination of the sun during that period.

We have seen that, in a day, the value of tidal acceleration varies quite a bit and can only be found from the exact location of the sun and the moon. Over about a day however the average tidal acceleration is mostly dependent on the declination of sun/moon and latitude. Noting that the factor $\cos(HA - 180°)$ in equation 6.1 runs 0 to 360° (about) in one day we can find an approximate amount by which $\frac{GM_{body} r}{d^3}$ will be multiplied to give the average tidal acceleration any day given pendulum latitude *Lat* and body declination *Dec*. Letting *x* stand in for $(HA - 180°)$ and assuming that Dec and

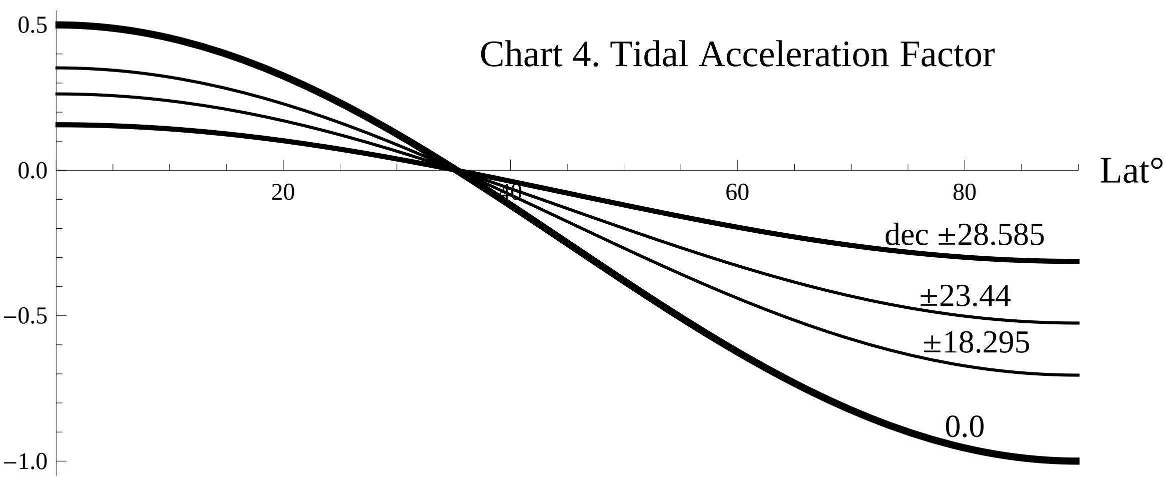
Lat are constant.

$$average\ daily\ tidal\ acceleration\ factor = \frac{1}{2\pi}\int_0^{2\pi}(3\cos^2 P - 1)\,dx \tag{6.2}$$

Next we can substutite equation 6.1 into 6.2 and carry out the integration and simplification and find a relation in declination and latitude only.

$$\frac{((3\cos(2\,Dec) - 1)(3\cos(2\,Lat) - 1))}{8} \tag{6.3}$$

Next we can plot for all latitudes and some sample declinations the curves that give this tidal acceleration factor:



Note that all these curves pass through zero at a single location which is $\arccos\sqrt{2/3}$ or 35.2644° matching the results in section 5. For declination 0°, range is maximum at 0.5 to -1.0 also matching the empirical results above.

Referring back to Chart 2 we see what is happening. At 35.2644° (we now know more exactly) average daily gravity is zero and during any day our example clock will have CTD ripple of about 1 or 2ms. In each month, the ripple will increase both below and above this latitude. The moon is seen in the full CTD curve but the variation is still small. The big effect is in the seasons as the sun take 3 months to move 23.44°. For example at 60° North or South this factor is -0.625 at declination 0 and -0.3283 at the tropics about 23.44°. So over these 3 months, the clock has time to accumulate this small change in *g*. This gives the much larger change at 60° of more than 20ms.

7. National Tidal Datum Epoch (1983-2001)

The people who do tides for a living (NOAA) have shown that a 19-year period is a good interval for evaluating the long term variations in tides. (see Reference 15) A figure in that document shows 20th century tides at Seattle, WA and shows clearly a roughly 19-year pattern. This period coresponds to the precession of the moon's orbit with respect to the plane of the earth's orbit which takes just under 19 years.

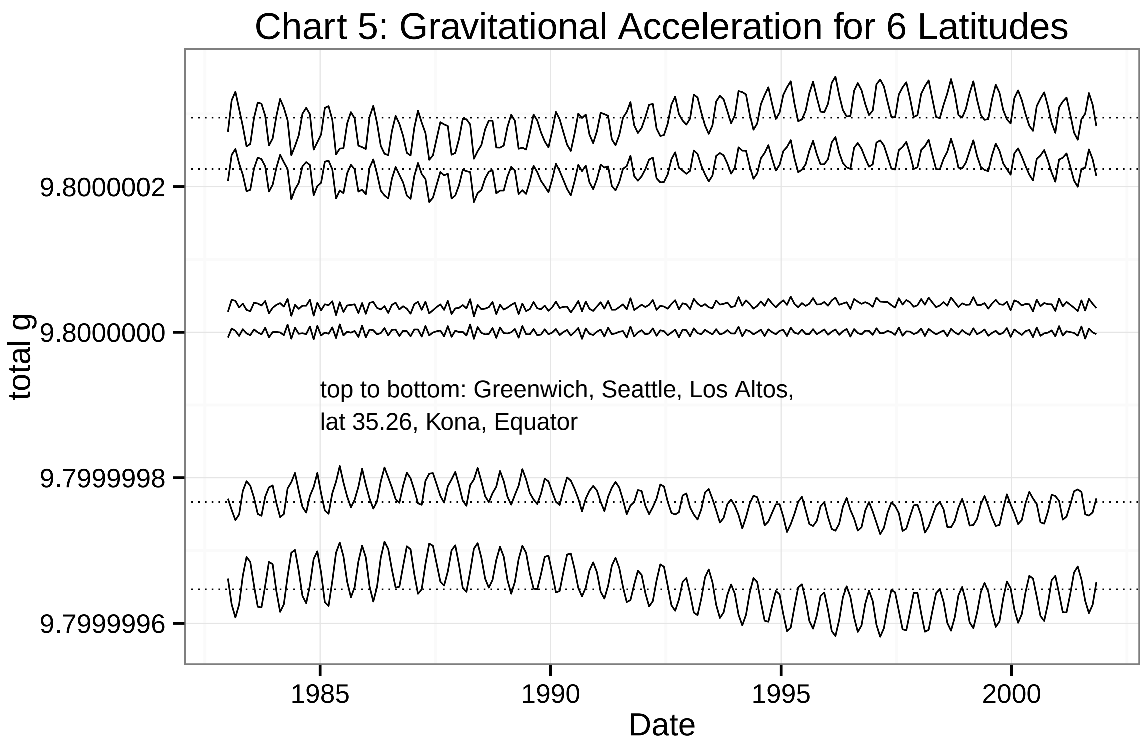


Chart 5: Gravitational Acceleration for 6 Latitudes
total g
9.8000002
9.8000000
9.7999998
9.7999996
top to bottom: Greenwich, Seattle, Los Altos,
lat 35.26, Kona, Equator
1985
1990
1995
2000
Date

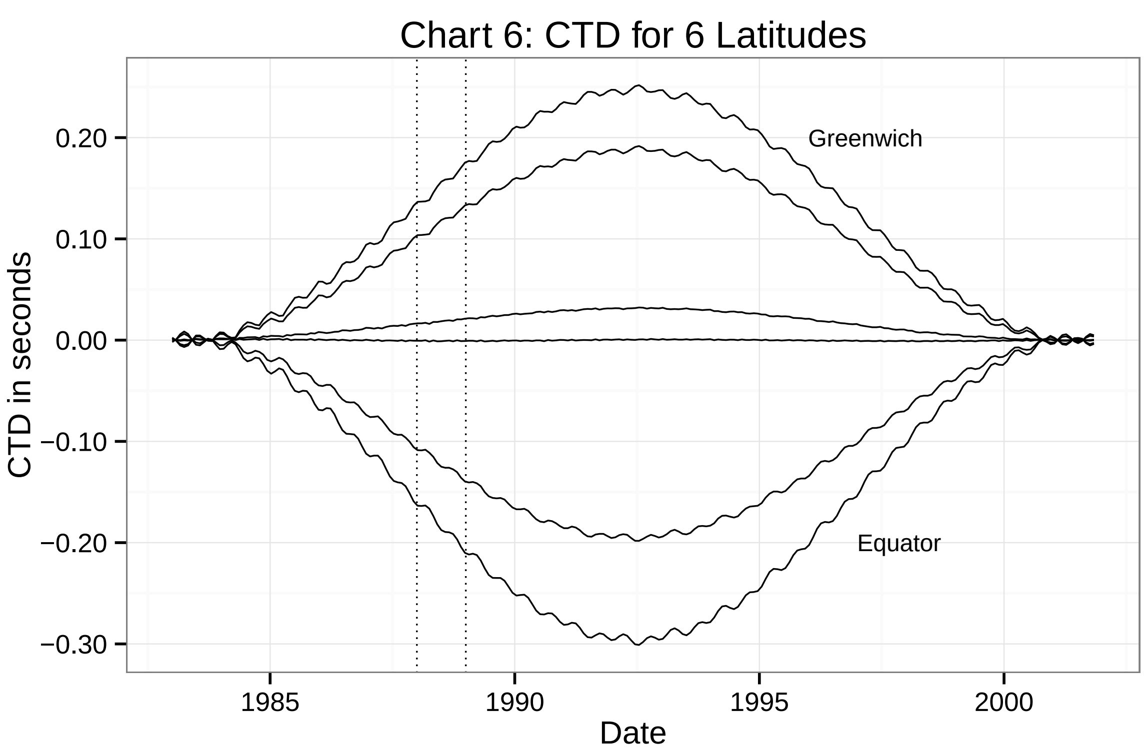
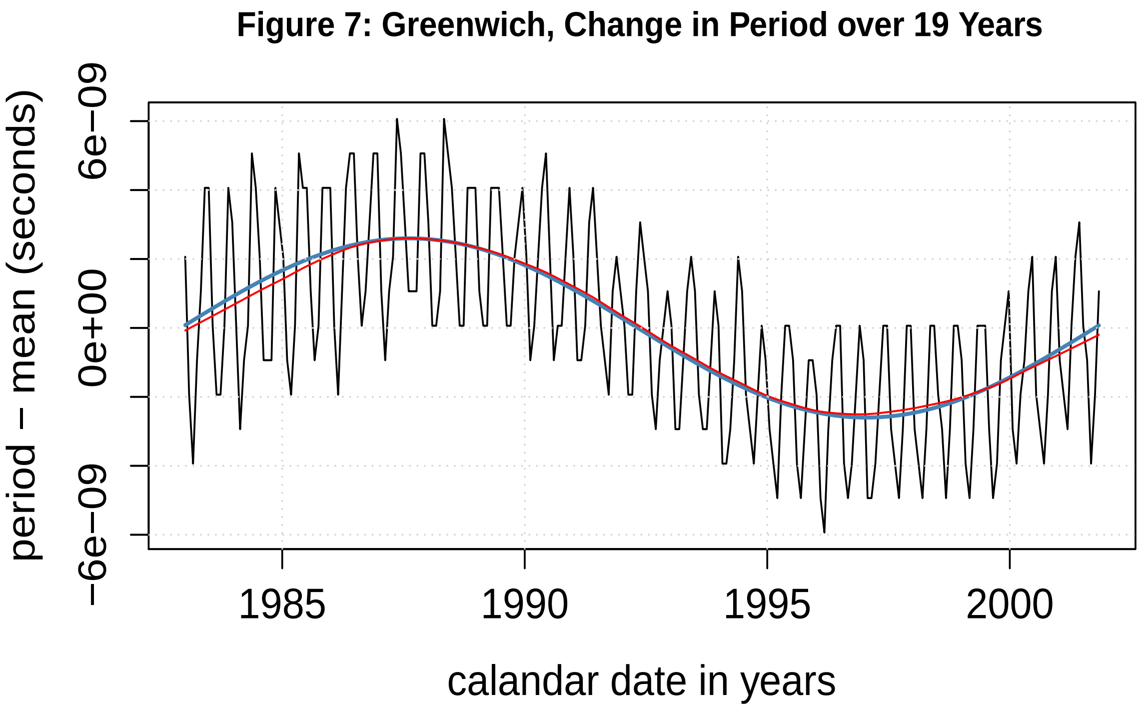


Chart 6: CTD for 6 Latitudes
CTD in seconds
0.20
0.10
0.00
−0.10
−0.20
−0.30
Greenwich
Equator
1985
1990
1995
2000
Date



Both the sun and moon have a large influence on tides. I examined a 19-year period starting in 1983 and continuing for 19 years throught 2001 to match the last complete NTDE. (see also Reference 16) By chance, data for chart 1 through 3 was taken just after the middle of the current epoch.

Using *tides*, I produced 6 files of hourly data from Jan 1983 through Dec 2001 for 6 model pendulums located at 51.5° (Greenwich UK), 41.6° (Seattle, WA), 37.4° (Los Altos, CA), 35.2644° (the still point), 19.6° (Kailua-Kona, HI), and finally at the equator. I wrote a small C program to average and summarize the results. I made an average of g over a 29.53 day month then created a CTD curve for each pendulum latitude.

Chart 5 shows the monthly value of total g for six latitudes. The northern most is Greenwich. Ripples are due to the sun through the seasons with two cycles per year as shown in chart 2. There is also a sine like wave with lower g for the first half of the epoch then rising in the second half. A dotted lines show average g. This corresponds to the precession of the moon's orbit over about 18.6 years. At latitude 35.2644° only a small ripple is seen from lunar/lunar variations. Below that point the phase of the ripples is shifted by 180° for both the lunar cycle and the seasonal cycle.

Above latitude 35.2644° the pendulums see higher average g and thus have a smaller period and appear to run fast compared to the one at the still point. Below is opposite and those appear to run slow. This matches the results before and is clearly predicted by the equations at the beginning of this paper.

Chart 6 is created to show how over this long 19 year cycle clocks above and below the still point actually accumulate a lot of error. At the equator the CTD will be 0.3 seconds slow. At Greenwich 0.25 fast. These curves are created by removing the average period of each pendulum to just show the variation over 19 years. Remember that only an absolutely stable pendulum could show this effect.

Figure 6 has the shape it does because the 1983-2001 epoch was chosen because it aligned with the 18.6 year period of the rotation of the nodes of the moon. If a 19 year plot had been made starting in a different year the shape of the curves would be phase shifted but always 1/2 cycle of a sine-like curve. The minimum and maximum deviation from zero would completely change but the range (max - min) will always be nearly the same - 0.25 seconds at Greenwich and 0.30 seconds at the equator.

Figure 7 isolates the period of a pendulum at Greenwich and subtracts the mean period of 2.00708989936 seconds. The chart shows that the overall range of values is ±6 nanoseconds. Two smooth curves are also shown. One is a smoothing function known as 'lowess' in statistics. The other is sine curve with amplitude 2.6 nanoseconds and a period of about 18.6 years. Both match quite closely.

Please refer again to Appendix A if necessary. The multiplying factor for the sum for the CTD is in this case 1,271,190 or $29.53 \times 24 \times 60 \times 60/\text{avgPeriod}$. For the first 9.3 years of this 18.6 year cycle the period is mostly longer than the mean. A quarter of the way into the 18.6 years even the deviation due to the sun is mostly above the mean. These account for the 0.25 second result. the average deviation from the mean in this case is about 1.3 nanoseconds which is consistant with the plots shown.

8. Summary

In this paper I have shown the mathematical model used to compute the effect of tidal gravity variations on a pendulum. Two methods of obtaining an ephemeris for the position of the sun and the moon for any location on earth are shown and checked for accuracy. CTD charts are used to show how integration of very small changes in g add up nanoseconds to milliseconds variations. Examples from a couple of days to 19 years are used to demonstrate the effect. An approximate analytical model is developed to show the latitude where tidal effects are constrained to daily effects. The mathematics has been checked with both the Mathmatica and Maxima symbolic algebra systems. This latitude is 35.2644° (north or south).

Few to no current or historical pendulums have had the long term stability to show the effects I have described. Perhaps future attempts can use the information here to remove one of the physical effects that change clock period.

9. Acknowledgements

During 2011 and 2015 Tom Van Baak and I exchanged numerious email messages and he greatly contributed to the form and content of the paper. In addition his translation of the *tides* program from Basic to C was a very great help. So many many thanks to Tom. Thanks also the Jim Hansen for his careful reading and corrections.

This paper, even more than my previous works, has used a vast array of computer software - some free and open source and some comercial fee based software. This time of special note is the R statistical package using RStudio as a frontend. Typesetting was done with *lout*. My thanks goes to all who spend their free time creating and maintaining these valuable systems.

10. Appendix A: Cumulative Time Deviation (CTD)

Generally we have a sequence of N values, pendulum period, from an average of k seconds of a pendulum operation. Call this sequence x_i where i goes from 1 to N. The *mean* period denoted by μ is:

$$\mu = \frac{1}{N}\sum_{i=1}^{N} x_i \tag{10.1}$$

The way a clock works is to add up each period by some means to show the time. A plot of this will just be a very slightly wavy straight line starting at 0 and going up and to the right. To see that waviness directly we have to subtract the mean from each period so the plot will generally stay near the x axis.

$$\sum (x_i - \mu) \tag{10.2}$$

To get CTD in seconds we have to multiply the sum by the number of pendulum cycles in k seconds, that is, k divided by period. The period could be either μ or x_i. In this paper I will use μ for simplicity because with tides the difference between μ and x_i is always very small. The correct formulations are then:

$$CTD_i = \frac{k}{\mu}\sum_{j=1}^{i} (x_j - \mu) \tag{10.3}$$

$$or\ \ CTD_i = \sum_{j=1}^{i} \frac{k}{x_j}(x_j - \mu) \tag{10.4}$$

Now we have at every instant of time in a test interval, (CTD_i) the difference, in seconds, of a clock affected by tidal gravity variations from one (magically) not affected.

The sensitivity of CTD as an analytical tool has features worth further discussion. First, with tides, period variations are all a sum of sinusoids with different amplitude, frequency and phase. Moreover they are periodic in various cycles. Water tides are predicted using *Harmonic Analysis* which is detailed in both reference 15 and 16. A CTD curve will always repeat, or nearly repeat, with the underlying periods. This means that the shape of the curve for the same periodicity which is started at various instants of time will be phase shifted from one started at another times. I did several experiments that showed the the range between the maximum and minimum values on the curve will always be the same for a fixed periodicity and sample interval.

Secondly, longer periodicities and longer sample intervals will show a larger range of deviation. In a day there aren’t enough cycles of the pendulum to add up to much. For a 19 year run the far larger number of cycles in the test and the long periodicity of the moon is the cause of much greater CTD range.

Thirdly, it does make a difference when a test is done because the configuration of the sun and moon are constantly changing. Any sailor will tell you that it makes a big difference if the tide is slack, max flood or max ebb. Here in the SanFrancisco bay some boats aren’t even fast enough to outrun the flow near the Golden Gate Bridge. The sine wave shown in Figure 7 shows the change in pendulum period - small and slow as it is. The derivative of sin(x) is cos(x). So the rate of change of pendulum period runs from a factor of 1 through 0 to -1 in each cycle. A test run at 1 will be different from one run at zero. A CTD curve will reflect this difference.

Finally, this form of analysis can be extended to other pendulum parameters such as temperature, barometric pressure, relative humidity, drive, and so on. CTD is a very strong low pass filter. I do find that it is useful to compare a CTD curve with one created in the same way for temperature or barometer.

11. Reference and Notes

1 Longman, I. M., *Formulas for computing the tidal acceleration due to the moon and the sun.* J. Geophys. Res., 64, 2351-2355, 1959. Source of equations used here and directly used in the tides.bas/c program.
http://www.ipgp.fr/files_lib/153_FormulasforComputingAccelerations.pdf (scanned)

2 Schureman, P., *A manual of the harmonic analysis and prediction of tides.* U.S. Coast and Geodetic Survey, Spec. Pub. 98, 1924 (revised in 1941 and 1958). Document cited by Longman. A pdf copy (1940) can be obtained:
http://docs.lib.noaa.gov/rescue/cgs_specpubs/QB275U35no981940.pdf
https://archive.org/stream/manualofharmonic00schu/manualofharmonic00schu_djvu.txt

3 Agnew, Duncan, Section 3.06 of Volume 3 of the 11 Volume set *Treatise on Geophysics* 2007 Elsevier B.V. pages 163 to 195. Duncan Agnew is/was at Scripps Institution of Oceanography, La Jolla, California, USA. A lovely pdf of that part is available at:
www.gps.caltech.edu/classes/ge167/file/agnew_treat_tide.pdf

4 Murphy, Tom, *Tidal Effects on Earth´s Surface* Feb 2, 2001. Also useful in understanding tidal effects. He notes that the work is an elaboration on F. D. Stacey´s *Physics of the Earth* which is still in print (check Amazon for example.) The Murphy pdf is:
http://physics.ucsd.edu/~tmurphy/apollo/doc/tides.pdf

5 A strange unattributed paper which is very helpful. These look like lecture notes and I wish I could see a video. Near the middle is a shorthand derivation of tidal acceleration for the sun or moon in about 6 lines. Given that the result is backed up in the previous, much harder to read, four references it is worth a look:
http://www.apl.ucl.ac.uk/lectures/3c37/3c37-6.html

6 This company makes a field gravitometer. The manual states that tidal correction is from the Longman paper and that "the gravimetric factor 1.16 in the Longman formula (16% increase over the amplitude of gravimetric tides of the rigid Earth)."
http://scintrexltd.com/
http://scintrexltd.com/dat/scintrexltd/files/CG-5%20Manual%20Ver%207.pdf

7 Clynch, James R., *Latitude and Longitudes in Geodesy* and *Radius of the Earth - Radii Used in Geodesy* February 2006. The relationship between latitude and earth radius isn't simple. These two documents clear things up.
www.oc.nps.edu/oc2902w/geodesy/latlons.pdf
www.oc.nps.edu/oc2902w/geodesy/radiigeo.pdf

8 Judson L. Ahern; Professor Emeritus, Geophysics; School of GeologyGeophysics, University of Oklahoma; Norman, OK 73019; (405) 325-4480
http://geophysics.ou.edu/

9 Sun and moon extreme distances:
http://en.wikipedia.org/wiki/Apsis
http://en.wikipedia.org/wiki/Moon

10 Horizons documentation:
http://ssd.jpl.nasa.gov/?horizons#doc

11 Peters, A *et. al.*, *High-precision gravity measurements using atom interferometry*, Metrologia, 2001, 38, 25-61. A sophisticated gravitometer built at Stanford University (37.428183, -122.173108). Figure 23 of that paper has a 9 day run which was used in test the software used here.
www.stanford.edu/group/chugroup/amo/amo.paper/metro-g.pdf

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http://en.wikipedia.org/wiki/Orbit_of_the_Moon
http://en.wikipedia.org/wiki/Lunar_standstill

15 Tidal Analysis and Prediction, NOAA Special Publication NOS CO-OPS 3, Bruce B. Parker, July 2007, 388pgs.
On page 42 Figure 2.18 'Variations in mean range of tide at Seattle, WA 1900 to 1996' shows the 18.6-year lunar nodal cycle.
http://tidesandcurrents.noaa.gov/publications/Tidal_Analysis_and_Predictions.pdf

16 Hicks, Steacy Dopp, Understanding Tides, December 2006, 83pgs.
Detailed but relatively easy to read overview of tides. Chapter 10 on datums describes how the National Tiadal Datum Epoch was chosen. Chapters 8 and 9 describe tidal harmonic analysis.
http://tidesandcurrents.noaa.gov/publications/Understanding_Tides_by_Steacy_finalFINAL11_30.pdf

By the time you read this paper some of these URL links will have died. First try a search on the title or author and see if it turned up somewhere else. I have copies of all of the .pdf files and can send them if needed.